EFFICIENT (PIECEWISE) LINEAR MINMAX APPROXIMATION OF DIGITAL SIGNALS

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ABSTRACT

In this work efficient geometric algorithms are provided for the linear approximation of digital signals under the uniform norm.
Given a set of n points $(x_i, y_i)_{i=1..n}$, with $x_i < x_j$ if $i < j$, we give a new method to find the optimum linear approximation in $O(n)$. Given also an error bound, we demonstrate how to construct in $O(n)$ a non continuous piecewise solution such that the number k of segments is optimal. Furthermore we show that for such number of segments, the solution that is l_∞ optimal can also be found in $O(n)$ provided that $n/k = O(1)$.

1. INTRODUCTION

The problem of approximating a discrete time signal s in n points x_i, $i = 1..n$ using the uniform norm basically consists on finding a function f pertaining to a class F of functions, such that

$$||s - f||_\infty = \max_{1 \le i \le n} |s(x_i) - f(x_i)|$$

is smallest than a given value δ or, in other cases, it is the smallest possible, that is

$$||s - f||_\infty = \min_{g \in F} ||s - g||_\infty$$

In this paper we aim at finding linear (single link) and noncontinuous piecewise linear (more links) approximations of signals.

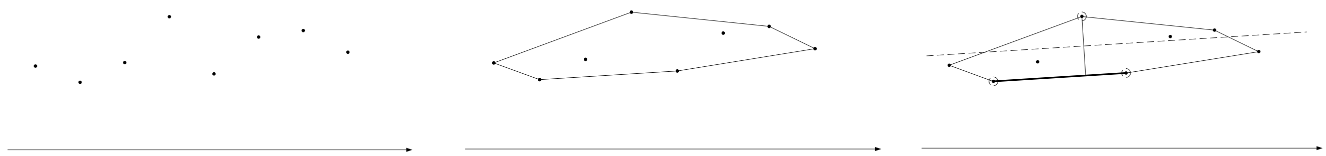
In the first case, it is known that the problem is equivalent to a linear programming problem in three variables that can be solved in $O(n)$ ([1], [2],[3]). We show instead a very efficient geometric method based on the properties of the set of points $S = \{(x_i, y_i\}_{i=1..n}$ (where $y_i = s(x_i)$); we compute the convex-hull Q of S and then we find a subset of three points of Q that univocally determines the solution. This leads to an algorithm that can take up to 1/8 of the time necessary for the linear programming solution.

In the second case, when an error bound δ is given, we show that it is possible to find, in $O(n)$ time, a noncontinuous piecewise linear function with k segments that approximate s with error $\epsilon \le \delta$ and such that the number of segments is optimal under this constraint; this means that there cannot exist a function with less than k segments approximating s with error $\epsilon' \le \delta$

Finally, we show that given again an error bound δ, after having found the minimum number of segments k and the k-link solution in the previous point, it is possible to find the optimal k-link solution in $O(n)$ if the length of the segments in the previous point is asymptotically bounded, which is true for the majority of practical situations.

2. SINGLE LINK SOLUTION

Given a set $S = \{x_i, y_i\}$ of n points that represents a discrete time signal, consider the convex hull Q of S.
Suppose for simplicity that Q has no pair of parallel sides (the opposite case can be handled similarly with a few some more considerations); then for every side $\overline{XY}$ of Q we can find one vertex Z of Q which is the most distant one $\overline{XY}$. We call Z *opposite vertex* to the side $\overline{XY}$.
Claim. There exists one side $\overline{AB}$ of Q such that its opposite vertex C satisfies $x_A < x_C < x_B$. The best l_∞ linear approximation of S is the line r parallel to $\overline{AB}$ and equidistant from $\overline{AB}$ and C.
Proof. Let p_i, $i = 1 \ldots k$, be the vertex of Q in counterclockwise order, with p_1 the left-most one. For clarity, let us add a point $p_{k+1} = p_1$ and let m, $m < k$, be the integer such that p_m is the right-most vertex. For $i = 1 \ldots k$, let l_i be the side $\overline{p_i p_{i+1}}$ and $v(l_i)$ be the opposite vertex to the side l_i. We say that $v(l_i)$ is x-internal to l_i if the vertical line through $v(l_i)$ cuts l_i.
We start by demonstrating that there exists at least one side l whose opposite vertex $v(l)$ is x-internal to it.
Suppose that every side l_i has its opposite vertex $v(l_i)$ that is not x-internal; then, clearly, $v(l_1)$ must be on the right of l_1 and $v(l_{m-1})$ must be on the left of l_{m-1}. So, there must exist an integer $j < m$ such that $v(l_{j-1})$ is on the right of



Signal samples — Convex-hull found — Solution found

Fig. 1. Steps of the geometric method for single link optimal solution

l_{j-1} and $v(l_j)$ is on the left of l_j. It is easy to see that p_j is the opposite vertex to every side between $v(l_{j-1})$ and $v(l_j)$; the vertical line through p_j must cut one of these sides and so there exists a side whose opposite vertex, p_j, is x-internal to it, so that the initial hypothesis was inconsistent.
Now, suppose we have three points A, B and C of Q such that C is the x-internal opposite vertex to the side $\overline{AB}$. For these three points the optimal linear approximation is easily proved to be the line r parallel to $\overline{AB}$ and equidistant from $\overline{AB}$ and C. The error committed by this line in approximating s at every x coordinate x_i is proportional to the distance of the point $s_i = (x_i, y_i)$ from the line; the way r has been selected [1] assures that A, B and C are the points of S mostly distante from r and so, the l_∞ approximation error of r is due to A, B and C. But for this three points r is optimal and so it is for the whole set S, as A, B, C are peculiar vertices of the convex hull.
Finally we have to show that there cannot exist another triplet of points A', B' and C' such that C' is x-internal to the side $\overline{A'B'}$. Supposing these three points exist, they should lead to an optimal solution r'. Calling $e(t; q_1, q_2, q_3)$ the error committed by the line t over the points q_1, q_2 and q_3 we should have

$$e(r'; A', B', C') \geq e(r'; A, B, C) \geq e(r; A, B, C) \quad (1)$$

because r' lead to its maximum error on A', B' and C', and r is optimum for A, B and C. But symmetrically we have

$$e(r, A, B, C) \geq e(r; A', B', C') \geq e(r'; A', B', C') \quad (2)$$

So the only possibility is that all these $\geq$ must be replaced by $=$ and, consequently, $r = r'$, which means that $\overline{AB}$ is parallel to $\overline{A'B'}$, contrarily to the hypothesis on Q. □

These considerations lead to the following efficient algorithm for finding the optimal line r (compare with fig. 1):

1. compute the convex-hull of the set S; since the points in S can be ranked by increasing x coordinate, such computation can be performed in $O(n)$ with Graham method ([4]), by incrementally adding points from left to right.
2. scan the M sides of the convex-hull computing their opposite vertex until the three points A, B and C are identified; with some tricks this can be performed in $O(M)$.
3. compute the solution of r in $O(1)$.

Compared to the linear programming solution this algorithm has many advantages. It is very easy to implement and generates a very compact code; all the computations in the construction of the convex-hull and the scanning of its vertices can be reduced to a scalar product operation of the type e=ac+cd. Moreover the only memory requirement (apart from the input sequence) is a vector containing the indices of the convex hull vertices which represent at most $n \times q$ bits, given a q bit digital sequence. One important thing to note is that, if we are working with q bits samples, all the computations in the first two steps of the method can be performed using only fixed point arithmetic. Thus the algorithm can be implemented without requiring a floating point unit.
Furthermore, since the optimal solution is based on selecting three particular points of S the solution for the whole set provides a mean to irregularly decimate the original sequence, with a controlled mean for reconstruction of the original sequence, as it is often required for example in near lossless data compression. So it provides a strategy for irregular sampling or a new mean for detection of relevant samples in a digital sequence.

3. PIECEWISE SOLUTION WITH ERROR BOUND

In this section we want to study a different problem and, in particular, we are interested to non-continuous piecewise linear approximations [2] .

For the sake of clarity, we suppose that the set S is characterized by $x_i = i$, $i = 1 \ldots n$ (even though the presented

[1]Consider that all the points of S lie in the strip of plane between the the line t passing through A and B and its parallel t' passing through C. The line r is exactly in the medium of this strip and the most distant points are the ones lying on t and t'.

[2]The proposed strategy for approximation shown in this section and the next one can be easily extended to any piecewise optimally minmax functional approximation. The difference simply lies in the procedure and computational complexity individually attached to each segment.

results hold in the case of non uniform samples). Given all the possible partitions of the domain, the problem of minmax approximating the sequence within each segment of the partition is a mere application of what explained in the previous section.
Given any piecewise minmax approximation g of the signal, we characterize it with an error $e(g)$, a number of segments $\nu(g)$ and a *partition* set $P(g) = \{p_i(g)\}_{i=1..\nu(g)-1}$ of values such that $p_i = m + 1/2$ if m is the last point of the i-th segment and $m+1$ is the first point of the $(i+1)$-th segment. Moreover, we set $p_0(g) = 1/2$ and $p_{\nu(g)}(g) = n+1/2$, and it is implicitly considered that the partition points p_i can only take values of the type $m + 1/2$ with $m \in \mathbb{N}$. All considered intervals are measured on a discrete half integer valued lattice. Thus we will identify the "approximation on the interval [3/2,7/2]" as "the one of locations 2 and 3"; similarly, by stating "the partition point p_i is in $]3/2, 7/2]$" is equivalent to say "$p_i \in \{5/2, 7/2\}$".

The problem to be solved is the following: given the set S of n samples of s and an error bound δ, we want to find a (non continuous) piecewise linear approximation g of s with error $e(g) \leq \delta$ such that the number $\nu(g)$ of segments used is the smallest possible. In general there are more solutions to the problem but, interestingly enough, two of them (not necessarily distinct) can be found in $O(n)$ in a simple way. We call them $\overrightarrow{g}$ (left-to-right solution) and $\overleftarrow{g}$ (right-to-left solution), the motivation will become obvious. It is implicit that $\overrightarrow{g}$ and $\overleftarrow{g}$ depend on the value δ. The algorithm that determines $\overrightarrow{g}$ is the following:

- Starting from the first point on the left, incrementally add points until the error bound of the associated 1-link solution δ is reached. This requires the minmax error to be computed at every step. However, given vertices A_m, B_m, C_m at step m and the convex-hull at step $m+1$, vertices A_{m+1}, B_{m+1} and C_{m+1} at step $m+1$ in $O(1)$ are straightforward to derive. Thus the total computational time remains simply proportional to the number sample points.
- When the error bound is reached, this determines the value $p_1(\overrightarrow{g})$. The process can now be started for a new segment, until $p_2(\overrightarrow{g})$ is found, i.e. when the error bound is reached a second time. The procedure is similarly carried out till all points in S have been dealt with.

Using the above construction process the number of segments used by $\overrightarrow{g}$ is optimal, which means that every piecewise function h approximating S with error $e(h) \leq \delta$ has a number of segments $\nu(h) \geq \nu(\overrightarrow{g})$.

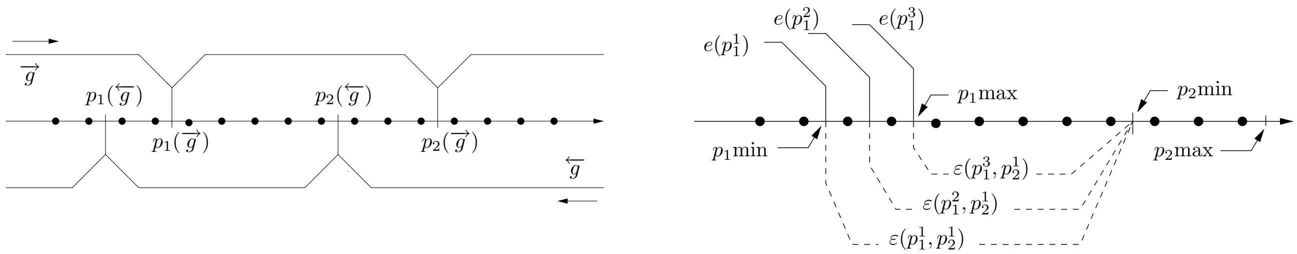
The way $p_1(\overrightarrow{g})$ was obtained when constructing the solution $\overrightarrow{g}$ implicitly says that it is not possible to approximate the interval $[1, p_1(\overrightarrow{g})+1]$ with a single segment (without exceeding the value of δ); h cannot be an exception and thus $P(h)$ must have a partition point $p_1(h)$ in the interval $[1, p_1(\overrightarrow{g})]$. Similarly it is not possible to approximate with a single segment the interval $[p_1(\overrightarrow{g}), p_2(\overrightarrow{g}) + 1]$, so that $P(h)$ must have at least another point $p_2(h)$ in $]p_1(\overrightarrow{g}), p_2(\overrightarrow{g})]$ as $p_1(h) \leq p_1(\overrightarrow{g})$. By iterating the argument, this proves by induction, that for $1 \leq i \leq \nu(\overrightarrow{g}) - 2$ there must exist a point of $P(h)$ in the interval $]p_i(\overrightarrow{g}), p_{i+1}(\overrightarrow{g})]$ and thus $\nu(h) \geq \nu(\overrightarrow{g})$.

By symmetry of the construction process, the same proof can be applied by replacing $\overrightarrow{g}$ with $\overleftarrow{g}$ while proceeding from right to left. This will ensure that $\nu(h) \geq \nu(\overleftarrow{g})$. Now, setting $h = \overleftarrow{g}$ has lead to $\nu(\overleftarrow{g}) \geq \nu(\overrightarrow{g})$ while setting $h = \overrightarrow{g}$ has lead to $\nu(\overrightarrow{g}) \geq \nu(\overleftarrow{g})$.
This proves that $\nu(\overrightarrow{g}) = \nu(\overleftarrow{g}) = k$ and that this number k of segments defines the minimum of the piecewise minmax optimal solution.

4. OPTIMAL K-LINK SOLUTION

We have shown in the preceding section how to find a piecewise solution with optimal number of segments in $O(n)$ when an error bound is given. Once we know the minimum number k of segments, we can ask for the solution in k segments with minimum error. The general problem of finding the optimal solution given the number of segments is solvable in $O(n \log n)$, as shown by Goodrich in [5], if one is interested in continuous approximations. In our case, on the contrary, we are looking for noncontinuous approximations, but we can take advantage of the fact that we already know two solutions ($\overrightarrow{g}$ and $\overleftarrow{g}$) with k segments. In fact we will show that these two solutions provide two partition sets $P(\overrightarrow{g})$ and $P(\overleftarrow{g})$ that determine a sort of extremes of the partition set $P(h)$ of any possible k-link solution h. More precisely, if h satisfies $\nu(h) = k$ and $e(h) \leq \delta$ then $p_i(\overleftarrow{g}) \leq p_i(h) \leq p_i(\overrightarrow{g})$ for every $1 \leq i \leq k-1$.
If $e(h) \leq \delta$ we have already proven that there exists a point of $P(h)$ in $]p_i(\overrightarrow{g}), p_{i+1}(\overrightarrow{g})]$ for $0 \leq i \leq k-2$. If $\nu(h) = k$ then in each interval there is exactly one point, which has to be $p_{i+1}(h)$. This holds for $h = \overleftarrow{g}$ so that $p_i(\overrightarrow{g}) < p_{i+1}(\overleftarrow{g}) \leq p_{i+1}(\overrightarrow{g})$. By symmetry we can say that if $e(h) \leq \delta$ and $\nu(h) = k$ there is exactly one point $p_i(h)$ in $[p_i(\overleftarrow{g}), p_{i+1}(\overleftarrow{g})[$ for $1 \leq i \leq k-1$ and, for $h = \overrightarrow{g}$ we obtain $p_i(\overleftarrow{g}) \leq p_i(\overrightarrow{g}) < p_{i+1}(\overleftarrow{g})$. By combining these inequalities we reach the result that if h is a optimal k-link solution then $p_i(\overleftarrow{g}) \leq p_i(h) \leq p_i(\overrightarrow{g})$ for every $1 \leq i \leq k-1$.

The above consideration provides a very important property of the possible partitions of the domain to obtain a k-link solution.
We now show how to find the solution f that minimizes the maximum error in $O(n)$. The idea is to proceed from left to right and find at every location of the candidate i-th partition point the partition point p_{i-1} that minimizes the error



Partition of the domain for $\overrightarrow{g}$ (up) and $\overleftarrow{g}$ (down). Possible positions for p_1 and p_2, with relative link errors.

Fig. 2. Finding the optimum partition set P. By computing $\overrightarrow{g}$ and $\overleftarrow{g}$ we first find the possible ranges for the values p_i. Then, for every possible value of p_{i+1} we compute the value of p_i that minimizes the maximum error on the corresponding link; at the last step p_{k-1} will be determined and consequently, every p_i.

for all $x < p_i$ (which will be indicated with $e(p_i)$).

Let us call $w_i = p_i(\overrightarrow{g}) - p_i(\overleftarrow{g}) + 1$; we know that for the first point, p_1, we can choose between w_1 different positions, that is $p_1(\overleftarrow{g})$, $p_1(\overleftarrow{g}) + 1$, $p_1(\overleftarrow{g}) + 2$, ... $p_1(\overrightarrow{g})$. For every candidate $p_1^l = p_1(\overleftarrow{g}) + l - 1$, $l = 1 \dots w_1$, we compute the approximation error of the first segment, as we did when finding $\overrightarrow{g}$, and we denote the corresponding values $e(p_1^1)$, $e(p_1^2) \dots e(p_1^{w_1})$, respectively.

Now, for every possible position p_2^r, $r = 1 \dots w_2$ of p_2 and for every possible position p_1^l of p_1, we compute the error $\varepsilon(p_1^l, p_2^r)$ of approximating the interval $[p_1^l, p_2^r]$ with one segment.

For every point p_2^r we compute the value $e(p_2^r)$ given by

$$e(p_2^r) = \min_{1 \le l \le w_1} \left(\max \left(e(p_1^l), \varepsilon(p_1^l, p_2^r) \right) \right) \qquad (3)$$

that is the minimum error that we can induce in the domain $x < p_2^r$ if we choose the second partition point $p_2 = p_2^r$.

The value of l for which the minimum is reached in (3) determines the best choice for p_1 given that $p_2 = p_2^r$. By iterating the process (as $e(p_2^1)$, $e(p_2^2)$, ... are known) we can determine the best choice for p_2 for all possible candidate positions of p_3 and so on. At the last stage we will find the best position of p_{k-1} given that $p_k = n + 1/2$. Now by backpropagation, we can determine the optimum p_{k-2}, p_{k-3}, ..., p_1.

Let us now assess the computational complexity of the described procedure. At step i, $w_i \cdot w_{i-1}$ errors need to be estimated. If we move p_i in the inner cycle and p_{i-1} in the outer one we can assume the number of computations be less than $w_{i-1} \cdot (p_i(\overrightarrow{g}) - p_i(\overleftarrow{g}))$. By calling l_i the length $p_i(\overrightarrow{g}) - p_{i-1}(\overrightarrow{g})$ of the i-th segment of $\overrightarrow{g}$, we have $w_{i-1} \cdot (p_i(\overrightarrow{g}) - p_i(\overleftarrow{g})) < l_{i-1} \cdot (l_{i-1} + l_i)$.

If we consider that the maximum length of the segments is asymptotically constant in the number of points, which actually makes sense in many practical situations (consider for example electrocardiographic signals), then the number of operations for every p_i is bounded by a constant and so the complexity of the algorithm is $O(n)$.

5. CONCLUSION

We have presented a new geometric algorithm for the linear approximation of signals in the l^∞ norm. The algorithm is very efficient both in terms of its computational efficiency, memory usage and ease of implementation. Given an error bound δ, we have shown how to find a piecewise linear approximation of a signal with minimum number of segments k in $O(n)$. Finally, we have shown how to minimize the error for the same number k of segments in $O(n)$.

6. REFERENCES

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