Constant Compositions in the Sphere Packing Bound for Classical-Quantum Channels

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Classical-Quantum Channels: Notation

- $\mathcal{X}$ discrete set of input symbols
- Associated (finite dimentional) density operators $S_x, x \in \mathcal{X}$
- n-fold product channel:

$$\begin{array}{ccc} \text{Codeword} & & \text{Product State} \\ \boldsymbol{x} = (x_1, x_2, \ldots, x_n) & \mapsto & \boldsymbol{S_x} = S_{x_1} \otimes S_{x_2} \cdots \otimes S_{x_n} \end{array}$$

- Code: a set of M codewords $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\} \subset \mathcal{X}^n$
- Quantum decision scheme: a set of M positive operators $\boldsymbol{\Pi} = \{\Pi_1, \Pi_2, \ldots, \Pi_M\}$ with $\sum_i \Pi_i \leq \mathbb{1}$
- Rate

$$R = \frac{\log M}{n}$$

Classical-Quantum Channels: Notation

- Probability of error

$$\mathsf{P}_{\mathrm{e}|m} = 1 - \mathrm{Tr}(\Pi_m \boldsymbol{S}_{\boldsymbol{x}_m}), \qquad \mathsf{P}_{\mathrm{e},\max} = \max_m \mathsf{P}_{\mathrm{e}|m}$$

- Optimal probability of error

$$\mathsf{P}_{\mathrm{e},\max}^{(n)}(R) = \min_{\{\boldsymbol{x}_m\},\boldsymbol{\Pi}} \mathsf{P}_{\mathrm{e},\max}$$

(min. over coding schemes of block-length n and rate at least R).

- Reliability function

$$E(R) = \limsup_{n\to\infty} -\frac{1}{n} \log \mathsf{P}_{\mathrm{e},\max}^{(n)}(R) \qquad (\text{i.e., } \mathsf{P}_{\mathrm{e}} \approx e^{-nE(R)})$$

- Zero-error capacity

$$C_0 = \sup\{R : \mathsf{P}_{\mathrm{e},\max}^{(n)}(R) = 0 \text{ for some } n\} \tag{1}$$

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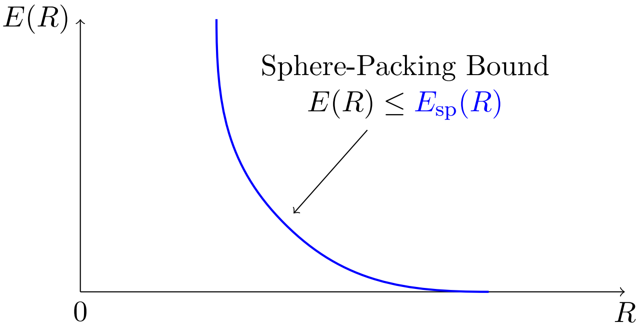

Dalai, ISIT 2012 (extending S-G-B 1967)

$$E_{\mathrm{sp}}(R) = \sup_{\rho \geq 0} \max_{P} \left(\overbrace{\left[-\log \mathrm{Tr} \left(\sum_x P(x) S_x^{1/(1+\rho)} \right)^{1+\rho} \right]}^{E_0(\rho,P)} - \rho R \right)$$

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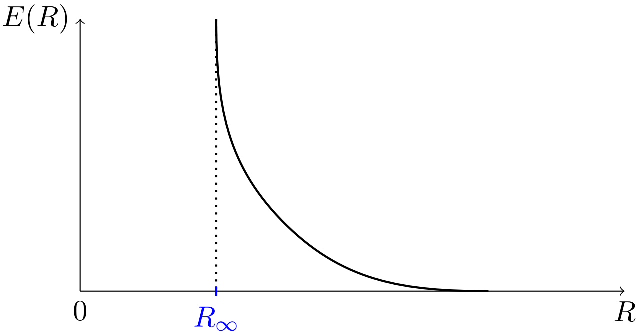

Upper bound on the zero-error capacity C_0

$$C_0 \le R_\infty$$

where

$$R_\infty = \min_F \max_x \log \frac{1}{\mathrm{Tr}\left(S_x^0 F\right)}$$

Sphere Packing and the Lovász theta Function

- Given a confusability graph G, we can optimize over channels

$$\vartheta_{\mathrm{sp}}(G) = \min_{\{S_x\}} R_\infty(\{S_x\})$$

over $\{S_x\}$ with confusability graph G.

- Note: consider pure S_x and F

$$\begin{aligned}\vartheta_{\mathrm{sp}}(G) &= \min_{\{S_x\}} \min_{F} \max_{x} \log \frac{1}{\mathrm{Tr}\left(S_x^0 F\right)} \\ &\leq \min_{\{u_x\}} \min_{f} \max_{x} \log \frac{1}{|\langle u_x|f\rangle|^2} \\ &= \vartheta(G)\end{aligned}$$

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- Fact:

$$\vartheta_{\mathrm{sp}}(G) = \vartheta(G), \tag{2}$$

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Constant composition (c.c.) codes

- All codewords with the same composition P_n
- $\mathsf{P}_{\text{e,max}}^{(n)}(R, P_n)$ as before but with c.c. P_n
- Reliability function with c.c.

$$E(R,P) = \limsup_{n\to\infty} -\frac{1}{n}\log \mathsf{P}_{\text{e,max}}^{(n)}(R,P_n) \qquad \text{(where } P_n \to P\text{)}$$

- Zero-error capacity

$$C_0(P) = \inf\{R : E(R,P) < \infty\}$$

- Questions
  - Bounds on $E(R,P)$?
  - Bounds on $C_0(P)$?

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$$E_{\mathrm{sp}}(R) = \max_P \sup_{\rho \geq 0} \left[-\log \mathrm{Tr} \left(\sum_x P(x) S_x^{1/(1+\rho)} \right)^{1+\rho} - \rho R \right]$$

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Only for commuting S_x

$E_{\text{sp}}^{\text{cc}}(R, P)$ for non commuting S_x

Theorem

For $R > 0$, $0 < \varepsilon < R$,

$$E(R, P) \leq E_{sp}^{cc}(R - \varepsilon, P),$$

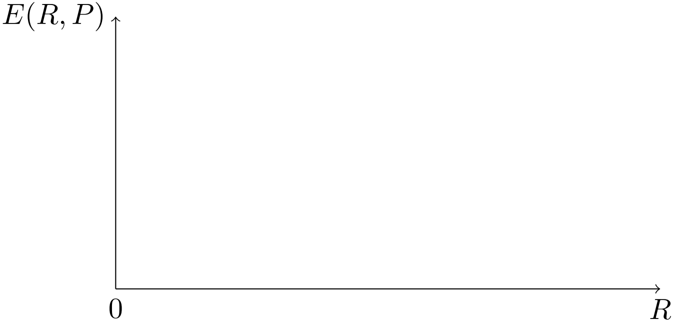
where

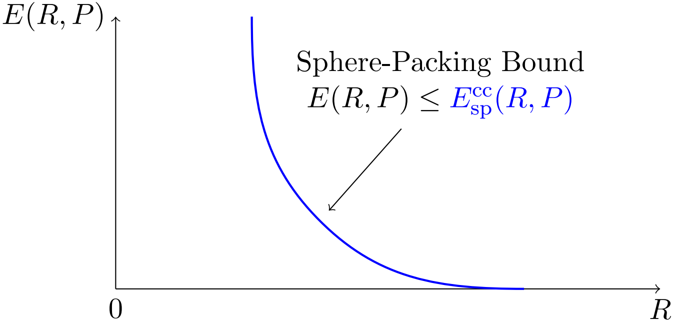
$$E_{sp}^{cc}(R, P) = \sup_{\rho \geq 0} \left[E_0^{cc}(\rho, P) - \rho R \right],$$

$$E_0^{cc}(\rho, P) = \min_F \left[-(1+\rho) \sum_x P(x) \log \operatorname{Tr}\left(S_x^{\frac{1}{1+\rho}} F^{\frac{\rho}{1+\rho}}\right) \right].$$

the minimum being over all density operators F.

Sphere Packing for Constant Composition Codes





This paper

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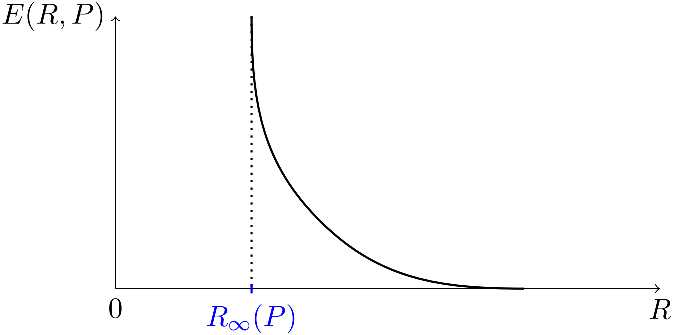

Then

$$C_0(P) \leq R_\infty(P)$$

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Connention with Marton's bound on C_0

- Optimize over channels with confusability graph G

$$\vartheta_{\text{sp}}(G,P) = \min_{\{S_x\}} R_\infty(\{S_x\},P) = \min_{\{S_x\}} \min_F \sum_x P(x) \log \frac{1}{\text{Tr}(S_x^0 F)},$$

- Using only pure states we get

$$\vartheta_{\text{sp}}(G,P) \leq \min_{\{u_x\}} \min_f \sum_x P(x) \log \frac{1}{|\langle u_x|f\rangle|^2} = \vartheta(G,P)$$

where $\vartheta(G,P)$ is Marton's variation of the Lovász theta function.

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- Bounding C_0:

$$C_0 \le \vartheta(G) \qquad \text{(Lovász)} \tag{3}$$

and

$$\begin{aligned} C_0 &= \max_P C_0(P) \\ &\le \max_P \vartheta(G,P) \qquad \text{(Marton)} \end{aligned}$$

- Marton: "it is easily seen that" $\max_P \vartheta(G,P) = \vartheta(G)$
  ... perhaps not really easily

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$$\max_P \vartheta(G,P) = \vartheta(G) \tag{4}$$

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Conditional Constant Composition

- Assume we have a finite set of states $\mathcal{A}$ and different channels $\{\mathcal{C}_a\}_{a\in\mathcal{A}}$
- Transmission governed by a sequence of states $\boldsymbol{a} = (a_1, \ldots, a_n)$ with composition P
- In a state $a \in \mathcal{A}$, channel $\mathcal{C}_a$ is used (hence, channel a is used $nP(a)$ times)
- Codewords with conditional composition V given $\boldsymbol{a}$ (each codeword has symbol x in a fraction $V(x|a)$ of the $nP(a)$ positions where $a_i = a$)
- For this scenario we can define an error exponent

$$E(\{\mathcal{C}_a\}, R, P, V)$$

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- In a state $a \in \mathcal{A}$, channel $\mathcal{C}_a$ is used (hence, channel a is used $nP(a)$ times)
- Codewords with conditional composition V given $\boldsymbol{a}$ (each codeword has symbol x in a fraction $V(x|a)$ of the $nP(a)$ positions where $a_i = a$)
- For this scenario we can define an error exponent

$$E(\{\mathcal{C}_a\}, R, P, V)$$

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Conditional Sphere Packing

Theorem

We have

$$E(\{\mathcal{C}_a\}, R, P, V) \leq E_{sp}^{cc}(\{\mathcal{C}_a\}, R - \varepsilon, P, V), \tag{5}$$

where $E_{sp}^{cc}(\{\mathcal{C}_a\}, R, P, V)$ *is defined by*

$$E_{sp}^{cc}(\{\mathcal{C}_a\}, R, P, V) = \sup_{\rho \geq 0} \left[E_0^{cc}(\{\mathcal{C}_a\}, \rho, P, V) - \rho R \right], \tag{6}$$

$$E_0^{cc}(\{\mathcal{C}_a\}, \rho, P, V) = \sum_a P(a) E_0^{cc}(\mathcal{C}_a, \rho, V(\cdot|a)). \tag{7}$$

and $E_0^{cc}(\mathcal{C}_a, \rho, V(\cdot|a))$ *is the coefficient* E_0^{cc} *of the sphere packing bound for channel* $\mathcal{C}_a$ *with composition* $V(\cdot|a)$*.*

An application

- Take a code with $M = e^{nR}$ codewords of composition P
- Given a matrix V, we can identify a subset $\mathcal{T}$ of codewords with conditional composition V from a particular sequence $\bar{x}$ where $|\mathcal{T}| = e^{n(R-I(P,V)-o(1))} = e^{nR'}$.
- We can use the sphere packing bound at rate R' for these codewords when used on auxiliary channels; channel $\tilde{\mathcal{C}}_{\bar{x}_i}$ at time instant i.
- Example for pure states: if $S_x = |\psi_x\rangle\langle\psi_x|$, for some $\rho \geq 1$ choose $\tilde{S}_{\bar{x}_i,x} = |\tilde{\psi}_{\bar{x}_i,x}\rangle\langle\tilde{\psi}_{\bar{x}_i,x}|$ such that $|\langle\tilde{\psi}_{\bar{x}_i,x}|\tilde{\psi}_{\bar{x}_i,x'}\rangle| \leq |\langle\psi_x|\psi_{x'}\rangle|^{1/\rho}$.

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Extension of the Elias bound

Theorem

We have $E(R,P) \le E_{spu}^{cc}(R,P)$ *where*

$$E_{spu}^{cc}(R,P) = \inf \rho[E_{sp}^{cc}(\{\tilde{\mathcal{C}}_a\}, R - I(P,V) - \varepsilon, P, V) + R - I(P,V)],$$

over $\varepsilon > 0$, $\rho \ge 1$, $\{\tilde{\mathcal{C}}_a\}$, V.

- It includes the previous "umbrella" bounds (Dalai ISIT 2013, TIT 2013)
- It includes the Elias bound and its extension to the Bhattacharyya distance of arbitrary channels (Dalai ISIT 2014)

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