An Elias Bound on the Bhattacharyya Distance of Codes for Channels with a Zero-Error Capacity

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Elias Bound for Binary Codes

- **Alphabet:** $\mathcal{X} = \{0, 1\}$
- **Sequence:** $x = (x_1, x_2, \ldots, x_n), \; x_i \in \mathcal{X}$
- **Hamming Distance:**
  \[
  d_H(x, x') = \sum_{i=1}^{n} d_H(x_i, x'_i), \quad \text{where } d_H(x, x') = \begin{cases} 
  0 & \text{if } x = x' \\
  1 & \text{if } x \neq x'
  \end{cases}
  \]
- **Code:** $\mathcal{C} = \{x_1, x_2, \ldots, x_M\}, \; x_i \in \mathcal{X}^n$.
- **Rate:** $R = \log M/n$
- **Minimum Distance:**
  \[
  d_{\text{min}}(\mathcal{C}) = \min_{i \neq j} d_H(x_i, x_j)
  \]
**Elias Bound for Binary Codes**

- **Optimal Distance:**
  \[ d(R, n) = \max_{\mathcal{C}} d_{\min}(\mathcal{C}) \]
  over codes of length \( n \) and rate at least \( R \).

- **Asymptotic Distance:**
  \[ \delta^*(R) = \limsup_{n \to \infty} \frac{1}{n} d(R, n) \]

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**Theorem (Elias’ bound)**

If \( R = 1 - h(\lambda) \), \( \lambda \in (0, 1/2) \), then

\[ \delta^*(R) \leq 2\lambda(1 - \lambda) \]
Sketch of the proof

- $n$-bit sequences

\[ \sum_{x, x' \in C} d(x, x') \leq M(M-1) \sum_{x, x' \in \subset T \subset C} d(x, x') \]

This is not very good.

Use the Plotkin bound on $T$

\[ d_{\text{min}} \leq \frac{1}{|T|}(|T| - 1) \sum_{x, x' \in T} d(x, x') \]

= $n \sum_{i=1}^{\infty} \left( \sum_{x, x' \in T} d(x_i, x'_i) \right)$
Sketch of the proof

- $n$-bit sequences
- Codewords
Sketch of the proof

The Plotkin bound on $\mathcal{C}$ would say

$$d_{\min} \leq \frac{1}{M(M - 1)} \sum_{x, x' \in \mathcal{C}} d(x, x')$$

... this is not very good.
Sketch of the proof

\[ d_{\text{min}} \leq \frac{1}{M} \left( M - 1 \right) \sum_{x, x' \in C} d(x, x') \]

... this is not very good.

Use the Plotkin bound on \( T \)

\[ d_{\text{min}} \leq \frac{1}{|T|} \left( |T| - 1 \right) \sum_{x, x' \in T} d(x, x') = \sum_{i=1}^{n} \left( \sum_{x, x' \in T} d_{i}(x, x') \right) \]
Sketch of the proof
Sketch of the proof

Subset $\mathcal{T} \subset \mathcal{C}$. 

\[
\text{Bound in terms of } \sum_{x \in \mathcal{T}} d(x_i, \bar{x}_i) 
\]

Use constraint on \[
\sum_{x \in \mathcal{T}} d(x, \bar{x}) 
\]
Sketch of the proof

Subset $\mathcal{T} \subset \mathcal{C}$.

Use the Plotkin bound on $\mathcal{T}$

$$d_{\text{min}} \leq \frac{1}{|\mathcal{T}|(|\mathcal{T}| - 1)} \sum_{x,x' \in \mathcal{T}} d(x, x')$$

$$= \sum_{i=1}^{n} \left( \sum_{x,x' \in \mathcal{T}} d(x_i, x'_i) \right)$$
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Use constraint on $\sum_{x \in \mathcal{T}} d(x, \bar{x})$

Bound in terms of $\sum_{x \in \mathcal{T}} d(x_i, \bar{x}_i)$
Extensions

- **Berlekamp:**
  - $\mathcal{X} = \mathbb{Z}_q = \{0, 1, \ldots, q - 1\}$
  - Distances: Hamming, Lee

- **Piret**
  - $\mathcal{X} = \{e^{i2\pi k/q}\}_{k=0,...,q-1}$ (uniform points on the unit circle)
  - Distance: squared euclidean

- **Blahut:**
  - $\mathcal{X}$ input alphabet of a DMC $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$
  - Distance: Bhattacharyya distance

\[
  d_B(x, x') = -\log \sum_y \sqrt{W(y|x)W(y|x')} 
\]

- Assuming $d_B(x, x')$ finite $\forall x, x'$ (no zero-error capacity)
- Assuming that the matrix with entries $(\sum_y \sqrt{W(y|x)W(y|x')})^{1/\rho}$ is positive semidefinite for all $\rho \geq 1$
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Blahut’s bound

**Theorem (Blahut)**

\[ \delta^*(R) \leq \max_P \min_{V \in \mathcal{V}(P,R)} \sum_x P(x) \sum_{x_1,x_2} V(x_1|x)V(x_2|x)d_B(x_1,x_2) \]  

(1)

where

\[ \mathcal{V}(P,R) = \{ V : PV = P, I(P,V) \leq R \}. \]  

(2)
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where

\[
\mathcal{V}(P,R) = \{ V : PV = P, I(P,V) \leq R \}. \tag{2}
\]

**Comments**

- \( d_B(x, x') < \infty \) is needed for the Plotkin bound

\[
\frac{1}{|\mathcal{T}|(|\mathcal{T}| - 1)} \sum_{x, x' \in \mathcal{T}} d(x, x') \tag{3}
\]
Blahut’s bound

**Theorem (Blahut)**

\[
\delta^*(R) \leq \max_P \min_{V \in \mathcal{V}(P,R)} \sum_x P(x) \sum_{x_1, x_2} V(x_1 | x) V(x_2 | x) d_B(x_1, x_2) \tag{1}
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- That \(\{(\sum_y \sqrt{W(y|x)W(y|x')})^{1/\rho}\}\) be p.s. is asked because it gives concavity of \(\in \mathcal{V}(\cdot|x)\)
Comments

- The previous two conditions asked in Blahut’s bound imply that \( d_B \) is a squared euclidean distance
- The bound actually holds for all squared euclidean distances
- It includes Berlekamp’s and Piret’s bounds

Problem

- If the channel has a zero-error capacity \( d_B(x, x') = \infty \) for some \( x, x' \)
- The Plotkin bound is useless

⇒ Find a bound which holds for all discrete channels
**Blahut’s Bound**

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- Find a bound which holds for all discrete channels.

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$\implies$ Find a bound which holds for all discrete channels
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Problem

- If the channel has a zero-error capacity $d_B(x, x') = \infty$ for some $x, x'$.
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⇒ Find a bound which holds for all discrete channels.
Key Idea

Let $W(y|x)$ be the memoryless extension of the channel

- We want to upper bound

$$d_B(x_i, x_j) = -\log \sum_y \sqrt{W(y|x_i)W(y|x_j)}$$  \hspace{1cm} (4)

for some $x_i, x_j \in \mathcal{C}$

- Get rid of the log and replace the Plotkin step with a Lovász-like approach to lower bound

$$\min_{i \neq j} \sum_y \sqrt{W(y|x_i)W(y|x_j)}$$  \hspace{1cm} (5)

- We still identify a subset $\mathcal{T}$ but somehow use

$$d_{\min} \leq -\rho \log \left( \max_{x \in \mathcal{T}} \frac{1}{(|\mathcal{T}| - 1)} \sum_{x' \in \mathcal{T}\{x\}} e^{-d(x,x')/\rho} \right).$$  \hspace{1cm} (6)
Let \( W(y|x) \) be the memoryless extension of the channel

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\[
d_{\text{min}} \leq -\rho \log \left( \max_{x \in T} \frac{1}{(|T| - 1)} \sum_{x' \in T \setminus \{x\}} e^{-d(x, x')/\rho} \right). \] (6)
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Starting point: the \( \vartheta(\rho) \) function

Define

- \( \psi_x = \sqrt{W(\cdot | x)} \), so that \( d_B(x, x') = -\log \psi_x^\dagger \psi_{x'} \).
- Similarly \( \psi_x = \sqrt{W(\cdot | x)} \), so that \( d_B(x, x') = -\log \psi_x^\dagger \psi_{x'} \).

Collections of “tilted” vectors

\[
\Gamma(\rho) = \left\{ \{\tilde{\psi}_x\} : |\tilde{\psi}_x^\dagger \tilde{\psi}_{x'}| \leq (\psi_x^\dagger \psi_{x'})^{1/\rho} \right\}. \tag{7}
\]

An extension of the Lovász theta function

\[
\vartheta(\rho) = \min_{\{\tilde{\psi}_x\} \in \Gamma(\rho)} \min_f \max_x \log \frac{1}{|\tilde{\psi}_x^\dagger f|^2} \tag{8}
\]

**Theorem (Dalai, ISIT’2013)**

\[
\max_{m' \neq m} \psi_m^\dagger \psi_{m'} \geq \left( \frac{Me^{-n\vartheta(\rho)} - 1}{M - 1} \right)^{\rho}.
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Starting point: the $\vartheta(\rho)$ function

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Starting point: the $\vartheta(\rho)$ function

Bound on $\delta(R)$ (Lovász-like variation of Plotkin’s bound)

$$\max_{m' \neq m} \psi^\dagger_m \psi_{m'} \geq (\frac{Me^{-n\vartheta(\rho)} - 1}{M - 1})^\rho$$

$$\downarrow (n \to \infty)$$

$$\delta(R) \leq \rho \vartheta(\rho) \text{ for } R > \vartheta(\rho)$$  \hfill (9)

$$\downarrow (\rho \to \infty)$$

$$\delta(R) < \infty \text{ for } R > \vartheta \quad \text{(Lovász)}$$

Next steps
- Constant (conditional) composition codes
- Identify the subset $\mathcal{T}$
Constant composition codes

First step: constant composition codes

- Let $\delta(R, P)$ the minimum distance achievable with constant composition codes with composition tending to $P$

- Consider the constant-composition variation of $\vartheta(\rho)$ (cf. Marton)

$$
\vartheta(\rho, P) = \min_{\{\tilde{\psi}_x\} \in \Gamma(\rho)} \min_{f} \sum_{x} P(x) \log \frac{1}{|\tilde{\psi}_x f|^2} \tag{10}
$$

Then, for codes with composition $P$

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\max_{m' \neq m} \psi_m^\dagger \psi_{m'} \geq \left( \frac{Me^{-n\vartheta(\rho, P)} - 1}{M - 1} \right)^{\rho}
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**Bound on $\delta(R, P)$**

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\delta(R, P) \leq \rho \vartheta(\rho, P) \text{ for } R > \vartheta(\rho, P) \tag{11}
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Further extension

- For a distribution $P$ and a $|\mathcal{X}| \times |\mathcal{X}|$ stochastic matrix $V : \mathcal{X} \rightarrow \mathcal{X}$, define

$$\vartheta(\rho, P, V) = \sum_{x} P(x) \vartheta(\rho, V(\cdot | x))$$

(12)

- If we have a set $\mathcal{T}$ codewords $x_i$ with conditional composition $V$ given a fixed sequence $\bar{x}$, then

$$\max_{m \neq m'} \psi_{x_m}^\dagger \psi_{x_{m'}} \geq \left(\frac{|\mathcal{T}| e^{-n\vartheta(\rho, P, V)} - 1}{|\mathcal{T}| - 1}\right)^\rho.$$  

(13)

- We now only have to identify $\bar{x}$ and $\mathcal{T}$
- We want $|\mathcal{T}| \approx e^{n(\vartheta(\rho, P, V) + \varepsilon)}$
For a distribution $P$ and a $|\mathcal{X}| \times |\mathcal{X}|$ stochastic matrix $V : \mathcal{X} \rightarrow \mathcal{X}$, define

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If we have a set $\mathcal{T}$ codewords $x_i$ with conditional composition $V$ given a fixed sequence $\bar{x}$, then

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- We now only have to identify \( \bar{x} \) and \( \mathcal{T} \)

- We want \(|\mathcal{T}| \approx e^{n(\vartheta(\rho,P,V) + \varepsilon)}\)
**Lemma (Blahut)**

Given a code with \( M = e^{nR} \) codewords of composition \( P \), let \( V \) be stochastic matrix such that \( nP(x)V(x'|x) \) an integer, \( PV = P \), and

\[
R \geq I(P, V) + \vartheta(\rho, P, V) + \varepsilon. \tag{14}
\]

Then, there is at least one sequence \( \bar{x} \) of composition \( P \) (not necessarily a codeword) such that there are at least \( e^{n(\vartheta(\rho, P, V)+\varepsilon-o(1))} \) codewords with conditional composition \( V \) from \( \bar{x} \).

**Theorem (Main result)**

For given \( R, P \) and \( \rho \geq 0 \), let \( V \) be a \( |X| \times |X| \) stochastic matrix such that \( PV = P \). Then

\[
\delta^*(R, P) \leq \rho \vartheta(\rho, P, V), \text{ for } R > I(P, V) + \vartheta(\rho, P, V). \tag{15}
\]
Last step: define $\mathcal{T}$

**Lemma (Blahut)**

Given a code with $M = e^{nR}$ codewords of composition $P$, let $V$ be stochastic matrix such that $nP(x)V(x'|x)$ an integer, $PV = P$, and

$$R \geq I(P, V) + \vartheta(\rho, P, V) + \varepsilon. \quad (14)$$

Then, there is at least one sequence $\bar{x}$ of composition $P$ (not necessarily a codeword) such that there are at least $e^{n(\vartheta(\rho, P, V) + \varepsilon - o(1))}$ codewords with conditional composition $V$ from $\bar{x}$.

**Theorem (Main result)**

For given $R$, $P$ and $\rho \geq 0$, let $V$ be a $|X| \times |X|$ stochastic matrix such that $PV = P$. Then

$$\delta^*(R, P) \leq \rho \vartheta(\rho, P, V), \text{ for } R > I(P, V) + \vartheta(\rho, P, V). \quad (15)$$
Comments$^1$

- The bound can be formulated for general (not necessarily Bhattacharyya) distances
- For squared Euclidean distances, as $\rho \to \infty$ it gives back Blahut’s bound
- Blahut’s one includes Berlekamp’s and Piret’s
- Hence this bound includes all previous cases

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Remarks

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