An Elias Bound on the Bhattacharyya Distance of Codes for Channels with a Zero-Error Capacity

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International Symposium on Information Theory, 2014

Elias Bound for Binary Codes

- **Alphabet**: $\mathcal{X} = \{0, 1\}$
- **Sequence**: $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathcal{X}$
- **Hamming Distance**:

$$d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{x}') = \sum_{i=1}^{n} d_{\mathrm{H}}(x_i, x_i'), \quad \text{where } d_{\mathrm{H}}(x, x') = \begin{cases} 0 & \text{if } x = x' \\ 1 & \text{if } x \neq x' \end{cases}$$

- **Code**: $\boldsymbol{\mathcal{C}} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\}$, $\boldsymbol{x}_i \in \mathcal{X}^n$.
- **Rate**: $R = \log M / n$
- **Minimum Distance**:

$$d_{\min}(\boldsymbol{\mathcal{C}}) = \min_{i \neq j} d_{\mathrm{H}}(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

Elias Bound for Binary Codes

- **Optimal Distance**:

$$d(R,n) = \max_{\boldsymbol{\mathcal{C}}} d_{\min}(\boldsymbol{\mathcal{C}})$$

over codes of length n and rate at least R.

- **Asymptotic Distance**:

$$\delta^*(R) = \limsup_{n\to\infty} \frac{1}{n} d(R,n)$$

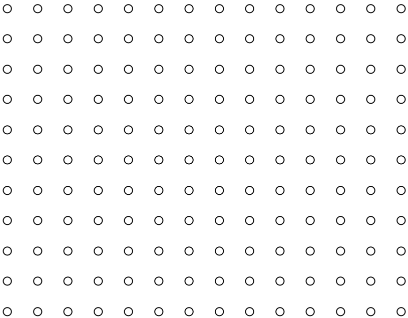
Theorem (Elias' bound)

If $R = 1 - h(\lambda)$, $\lambda \in (0, 1/2)$, *then*

$$\delta^*(R) \leq 2\lambda(1-\lambda)$$

Elias Bound for Binary Codes

Sketch of the proof



○ n-bit sequeces

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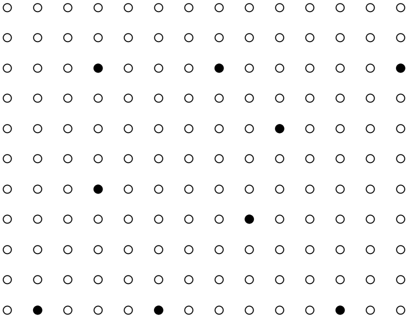

- ○ n-bit sequeces
- ● codewords

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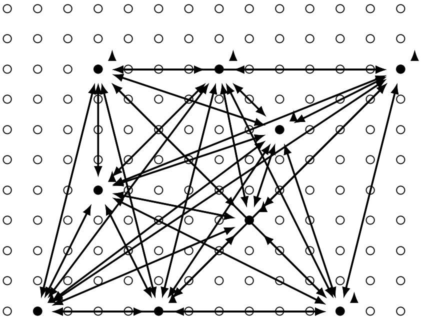

○ n-bit sequeces
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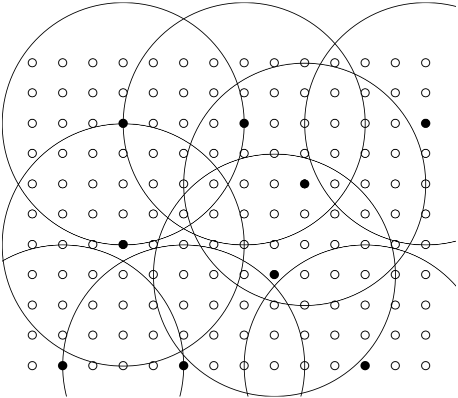
The Plotkin bound on $\mathcal{C}$ would say

$$d_{\min} \leq \frac{1}{M(M-1)} \sum_{\boldsymbol{x},\boldsymbol{x}' \in \mathcal{C}} d(\boldsymbol{x}, \boldsymbol{x}')$$

... this is not very good.

Sketch of the proof



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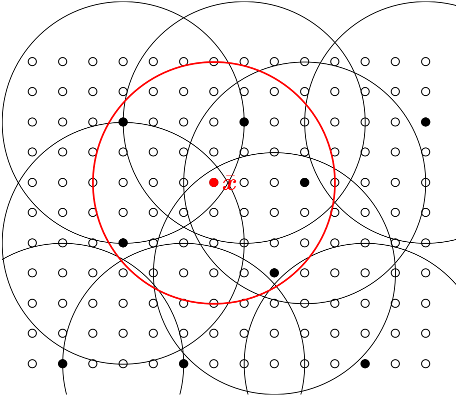
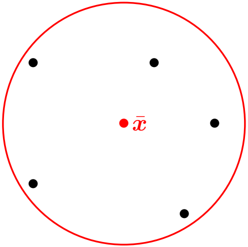


- ○ n-bit sequeces
- ● codewords
- ● critical sequence $\bar{\boldsymbol{x}}$

Sketch of the proof

Subset $\mathcal{T} \subset \boldsymbol{\mathcal{C}}$.



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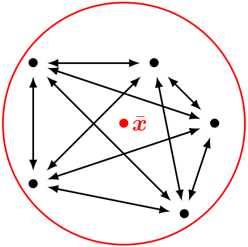

Use the Plotkin bound on $\mathcal{T}$

$$d_{\min} \leq \frac{1}{|\mathcal{T}|(|\mathcal{T}|-1)} \sum_{\boldsymbol{x},\boldsymbol{x}' \in \mathcal{T}} d(\boldsymbol{x}, \boldsymbol{x}')$$

$$= \sum_{i=1}^{n} \left(\sum_{\boldsymbol{x},\boldsymbol{x}' \in \mathcal{T}} d(x_i, x_i') \right)$$

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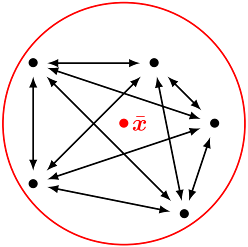

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$$= \sum_{i=1}^{n} \underbrace{\left(\sum_{\boldsymbol{x},\boldsymbol{x}' \in \mathcal{T}} d(x_i, x_i') \right)}$$

Use constraint on $\sum_{\boldsymbol{x} \in \mathcal{T}} d(\boldsymbol{x}, \bar{\boldsymbol{x}})$

Bound in terms of $\sum_{\boldsymbol{x} \in \mathcal{T}} d(x_i, \bar{x}_i)$

Extensions

- Berlekamp:
  - $\mathcal{X} = \mathbb{Z}_q = \{0, 1, \ldots, q-1\}$
  - Distances: Hamming, Lee
- Piret
  - $\mathcal{X} = \{e^{i2\pi k/q}\}_{k=0,\ldots,q-1}$ (uniform points on the unit circle)
  - Distance: squared euclidean
- Blahut:
  - $\mathcal{X}$ input alphabet of a DMC $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$
  - Distance: Bhattacharyya distance

$$d_{\mathrm{B}}(x, x') = -\log \sum_y \sqrt{W(y|x)W(y|x')}$$

  - Assuming $d_{\mathrm{B}}(x, x')$ finite $\forall x, x'$ (no zero-error capacity)
  - Assuming that the matrix with entries $(\sum_y \sqrt{W(y|x)W(y|x')})^{1/\rho}$ is positive semidefinite for all $\rho \geq 1$

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Blahut's bound

Theorem (Blahut)

$$\delta^*(R) \leq \max_P \min_{V \in \mathcal{V}(P,R)} \sum_x P(x) \sum_{x_1, x_2} V(x_1|x) V(x_2|x) d_B(x_1, x_2) \qquad (1)$$

where

$$\mathcal{V}(P, R) = \{V : PV = P, \, I(P, V) \leq R\}. \qquad (2)$$

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Comments

- $d_{\mathrm{B}}(x,x') < \infty$ is needed for the Plotkin bound

$$\frac{1}{|\mathcal{T}|(|\mathcal{T}|-1)} \sum_{\boldsymbol{x},\boldsymbol{x}' \in \mathcal{T}} d(\boldsymbol{x},\boldsymbol{x}') \qquad (3)$$

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- That $\{(\sum_y \sqrt{W(y|x)W(y|x')})^{1/\rho}\}$ be p.s. is asked because it gives concavity of ◯ in $V(\cdot|x)$

Comments

- The previous two conditions asked in Blahut's bound imply that d_{B} is a squared euclidean distance
- The bound actually holds for all squared euclidean distances
- It includes Berlekamp's and Piret's bounds

Problem

- If the channel has a zero-error capacity $d_{\mathrm{B}}(x, x') = \infty$ for some x, x'
- The Plotkin bound is useless

$\Longrightarrow$ Find a bound which holds for all discrete channels

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Key Idea

Let $\boldsymbol{W}(\boldsymbol{y}|\boldsymbol{x})$ be the memoryless extension of the channel

- We want to upper bound

$$d_{\mathrm{B}}(\boldsymbol{x}_i, \boldsymbol{x}_j) = -\log \sum_{\boldsymbol{y}} \sqrt{\boldsymbol{W}(\boldsymbol{y}|\boldsymbol{x}_i)\boldsymbol{W}(\boldsymbol{y}|\boldsymbol{x}_j)} \tag{4}$$

for some $\boldsymbol{x}_i, \boldsymbol{x}_j \in \boldsymbol{\mathcal{C}}$

- Get rid of the log and replace the Plotkin step with a Lovász-like approach to lower bound

$$\min_{i \neq j} \sum_{\boldsymbol{y}} \sqrt{\boldsymbol{W}(\boldsymbol{y}|\boldsymbol{x}_i)\boldsymbol{W}(\boldsymbol{y}|\boldsymbol{x}_j)} \tag{5}$$

- We still identify a subset $\mathcal{T}$ but somehow use

$$d_{\min} \leq -\rho \log \left(\max_{\boldsymbol{x} \in \mathcal{T}} \frac{1}{(|\mathcal{T}| - 1)} \sum_{\boldsymbol{x}' \in \mathcal{T} \setminus \{\boldsymbol{x}\}} e^{-d(\boldsymbol{x}, \boldsymbol{x}')/\rho} \right). \tag{6}$$

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Starting point: the $\vartheta(\rho)$ function

Define

- $\psi_x = \sqrt{W(\cdot|x)}$, so that $d_{\mathrm{B}}(x, x') = -\log \psi_x^\dagger \psi_{x'}$.
- Similarly $\boldsymbol{\psi_x} = \sqrt{\boldsymbol{W}(\cdot|\boldsymbol{x})}$, so that $d_{\mathrm{B}}(\boldsymbol{x}, \boldsymbol{x'}) = -\log \boldsymbol{\psi_x^\dagger \psi_{x'}}$.
- Collections of "tilted" vectors

$$\Gamma(\rho) = \left\{ \{\tilde{\psi}_x\} \,:\, |\tilde{\psi}_x^\dagger \tilde{\psi}_{x'}| \leq (\psi_x^\dagger \psi_{x'})^{1/\rho} \right\}. \tag{7}$$

- An extension of the Lovász theta function

$$\vartheta(\rho) = \min_{\{\tilde{\psi}_x\} \in \Gamma(\rho)} \min_f \max_x \log \frac{1}{|\tilde{\psi}_x^\dagger f|^2} \tag{8}$$

Theorem (Dalai, ISIT'2013)

$$\max_{m' \neq m} \boldsymbol{\psi}_m^\dagger \boldsymbol{\psi}_{m'} \geq \left(\frac{M e^{-n\vartheta(\rho)} - 1}{M - 1} \right)^\rho .$$

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Bound on $\delta(R)$ (Lovász-like variation of Plotkin's bound)

$$\max_{m' \neq m} \boldsymbol{\psi}_m^\dagger \boldsymbol{\psi}_{m'} \geq \left(\frac{M e^{-n\vartheta(\rho)} - 1}{M - 1} \right)^\rho$$

$$\Downarrow (n \to \infty)$$

$$\boxed{\delta(R) \leq \rho\vartheta(\rho) \text{ for } R > \vartheta(\rho)} \tag{9}$$

$$\Downarrow (\rho \to \infty)$$

$$\delta(R) < \infty \text{ for } R > \vartheta \qquad \text{(Lovász)}$$

Next steps

- Constant (conditional) composition codes
- Identify the subset $\mathcal{T}$

Constant composition codes

First step: constant composition codes

- Let $\delta(R,P)$ the minimum distance achievable with constant composition codes with composition tending to P
- Consider the constant-composition variation of $\vartheta(\rho)$ (cf. Marton)

$$\vartheta(\rho,P) = \min_{\{\tilde{\psi}_x\}\in\Gamma(\rho)} \min_f \sum_x P(x) \log \frac{1}{|\tilde{\psi}_x^\dagger f|^2} \tag{10}$$

Then, for codes with composition P

$$\max_{m'\neq m} \psi_m^\dagger \psi_{m'} \geq \left(\frac{Me^{-n\vartheta(\rho,P)}-1}{M-1}\right)^\rho .$$

Bound on $\delta(R,P)$

$$\delta(R,P) \leq \rho\vartheta(\rho,P) \text{ for } R > \vartheta(\rho,P) \tag{11}$$

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Constant *conditional* composition codes

Further extension

- For a distribution P and a $|\mathcal{X}| \times |\mathcal{X}|$ stochastic matrix $V : \mathcal{X} \to \mathcal{X}$, define

$$\vartheta(\rho, P, V) = \sum_x P(x)\vartheta(\rho, V(\cdot|x)) \tag{12}$$

- If we have a set $\mathcal{T}$ codewords $\boldsymbol{x}_i$ with conditional composition V given a fixed sequence $\bar{\boldsymbol{x}}$, then

$$\max_{m \neq m'} \psi^\dagger_{\boldsymbol{x}_m} \psi_{\boldsymbol{x}_{m'}} \geq \left(\frac{|\mathcal{T}| e^{-n\vartheta(\rho,P,V)} - 1}{|\mathcal{T}| - 1} \right)^\rho . \tag{13}$$

- We now only have to identify $\bar{\boldsymbol{x}}$ and $\mathcal{T}$
- We want $|\mathcal{T}| \approx e^{n(\vartheta(\rho,P,V)+\varepsilon)}$

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$$\max_{m \neq m'} \boldsymbol{\psi}^{\dagger}_{\boldsymbol{x}_m} \boldsymbol{\psi}_{\boldsymbol{x}_{m'}} \geq \left(\frac{|\mathcal{T}| e^{-n\vartheta(\rho,P,V)} - 1}{|\mathcal{T}| - 1} \right)^{\rho}. \tag{13}$$

- We now only have to identify $\bar{x}$ and $\mathcal{T}$
- We want $|\mathcal{T}| \approx e^{n(\vartheta(\rho,P,V)+\varepsilon)}$

Constant *conditional* composition codes

Further extension

- For a distribution P and a $|\mathcal{X}| \times |\mathcal{X}|$ stochastic matrix $V : \mathcal{X} \to \mathcal{X}$, define

$$\vartheta(\rho, P, V) = \sum_x P(x)\vartheta(\rho, V(\cdot|x)) \tag{12}$$

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Last step: define $\mathcal{T}$

Lemma (Blahut)

Given a code with $M = e^{nR}$ codewords of composition P, let V be stochastic matrix such that $nP(x)V(x'|x)$ an integer, $PV = P$, and

$$R \geq I(P,V) + \vartheta(\rho, P, V) + \varepsilon. \tag{14}$$

Then, there is at least one sequence $\bar{\boldsymbol{x}}$ of composition P (not necessarily a codeword) such that there are at least $e^{n(\vartheta(\rho,P,V)+\varepsilon-o(1))}$ codewords with conditional composition V from $\bar{\boldsymbol{x}}$.

Theorem (Main result)

For given R, P and $\rho \geq 0$, let V be a $|\mathcal{X}| \times |\mathcal{X}|$ stochastic matrix such that $PV = P$. Then

$$\delta^*(R,P) \leq \rho\vartheta(\rho,P,V), \text{ for } R > I(P,V) + \vartheta(\rho,P,V). \tag{15}$$

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Remarks

Comments[1]

- The bound can be formulated for general (not necessarily Bhattacharyya) distances
- For squared euclidean distances, as $\rho \to \infty$ it gives back Blahut's bound
- Blahut's one includes Berlekamp's and Piret's
- Hence this bound includes all previous cases

[1] M. Dalai, "Elias Bound for General Distances and Stable Sets in Edge-Weighted Graphs", *submitted* - arXiv:1406.1308.

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